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AUTHOR(S):

Giga, Mariko

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Some Operator Functions Implying Order Preserving Inequalities

日本医科大学 数学教室 儀我 真理子 (Mariko Giga)
(Department of Mathematics, Nippon Medical School)

This paper is a resume based on my talk at “Structure of operators and related recent topics” which has been held at RIMS on January 24, 2003 and also this is early announcement of [9].

As an application of our previous result [Theorem 1, 11], we show a simple proof of the following result:

If $A \geq B \geq C \geq 0$ with $A > 0$ and $B > 0$, then for each $t \in [0, 1]$, and $p \geq t$, the following (i) and (ii) hold for a fixed real number q and they are mutually equivalent:

(i) *if $q \geq 0$, then*

$$G_{p,q,t}(A, B, C, r, s) = A^{-\frac{r}{2}} \{A^{\frac{r}{2}} (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} A^{-\frac{r}{2}}$$

is decreasing function for $r \geq t$ and $s \geq 1$ such that $(p-t)s \geq q-t$.

(ii) *if $p \geq q$, then*

$$G_{p,q,t}(A, B, C, r, s) = A^{-\frac{r}{2}} \{A^{\frac{r}{2}} (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} A^{-\frac{r}{2}}$$

is decreasing function for $s \geq 1$ and $r \geq \max\{t, t-q\}$.

This result is further extension of our previous paper [Theorem 2, 11]. On the other hand, M.Uchiyama [17] shows the following interesting result

(iii) *If $A \geq B \geq C \geq 0$ with $B > 0$, then for each $t \in [0, 1]$ and $p \geq 1$,*

$$A^{1-t+r} \geq \{A^{\frac{r}{2}} (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \quad \text{holds for } r \geq t \text{ and } s \geq 1.$$

We show that (i) is equivalent to (iii), that is, follows from each other and also as an application of our previous result [Theorem 1, 11], we give a simple proof of M.Uchiyama's result [Theorem 3.4, 17].

1 Introduction.

A capital letter means a bounded linear operator on a Hilbert space.

Theorem L-H (Löwner-Heinz inequality) [13][15].

$$A \geq B \geq 0 \text{ ensures } A^\alpha \geq B^\alpha \text{ for all } \alpha \in [0, 1].$$

Theorem L-H is very useful, but the condition “ $\alpha \in [0, 1]$ ” is too restrictive to calculate operator inequalities, the following result has been obtained from this point of view.

Theorem F (Furuta inequality) [4].

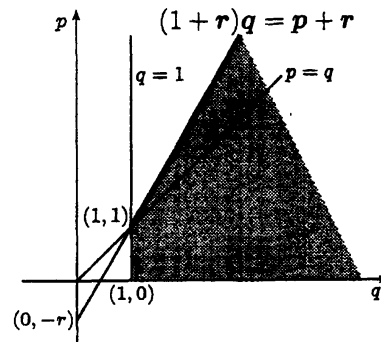
If $A \geq B \geq 0$, then for each $r \geq 0$,

$$(i) \quad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$(ii) \quad (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.



FIGURE

Alternative proofs are in [14][1] and one page proof in [5]. It is proved in [16] that The domain drawn for p, q and r in Figure is the best possible one for Theorem F. The following Theorem G is an extension of Theorem F.

Theorem G [6][2]. If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0, 1]$ and $p \geq 1$

$$A^{1-t+r} \geq \{A^{\frac{r}{2}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \text{ holds for } s \geq 1 \text{ and } r \geq t.$$

Very recently M.Uchiyama shows the following interesting extension of Theorem G.

Theorem U [17]. If $A \geq B \geq C \geq 0$ with $B > 0$, then for $t \in [0, 1]$ and $p \geq 1$

$$A^{1-t+r} \geq \{A^{\frac{r}{2}} (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \text{ holds for } s \geq 1 \text{ and } r \geq t.$$

We show that Theorem U is equivalent to (i) of Theorem 1 under below, that is, follows from each other and also as an application of our previous result [Theorem 1, 11], we give a simple proof of M.Uchiyama's result [Theorem 3.4, 17].

Operator Functions Implying Theorem U.

Theorem 1. *If $A \geq B \geq C \geq 0$ with $A > 0$ and $B > 0$, then for each $t \in [0, 1]$, and $p \geq t$, the following (i) and (ii) hold for a fixed real number q and they are mutually equivalent:*

(i) *if $q \geq 0$, then*

$$G_{p,q,t}(A, B, C, r, s) = A^{-\frac{r}{2}} \{A^{\frac{r}{2}} (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} A^{-\frac{r}{2}}$$

is decreasing function for $r \geq t$ and $s \geq 1$ such that $(p-t)s \geq q-t$.

(ii) *if $p \geq q$, then*

$$G_{p,q,t}(A, B, C, r, s) = A^{-\frac{r}{2}} \{A^{\frac{r}{2}} (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} A^{-\frac{r}{2}}$$

is decreasing function for $s \geq 1$ and $r \geq \max\{t, t-q\}$.

We need the following results to prove Theorem 1.

Theorem A [11]. *Let A and B be positive invertible operators on a Hilbert space satisfying*

$$A \geq (A^{\frac{1}{2}} B A^{\frac{1}{2}})^{\frac{\beta_0}{\alpha_0 + \beta_0}} \text{ for fixed } \alpha_0 \geq 0 \text{ and } \beta_0 \geq 0 \text{ with } \alpha_0 + \beta_0 > 0.$$

Then the following (i) and (ii) hold and they are mutually equivalent:

(i) *for any fixed $\delta \geq -\beta_0$,*

$$f(\lambda, \mu) = A^{-\frac{\mu}{2}} (A^{\frac{\mu}{2}} B^\lambda A^{\frac{\mu}{2}})^{\frac{\delta + \beta_0 \mu}{\alpha_0 \lambda + \beta_0 \mu}} A^{-\frac{\mu}{2}}$$

is decreasing function for $\mu \geq 1$ and $\lambda \geq 1$ such that $\alpha_0 \lambda \geq \delta$.

(ii) *for any fixed $\delta \leq \alpha_0$,*

$$f(\lambda, \mu) = A^{-\frac{\mu}{2}} (A^{\frac{\mu}{2}} B^\lambda A^{\frac{\mu}{2}})^{\frac{\delta + \beta_0 \mu}{\alpha_0 \lambda + \beta_0 \mu}} A^{-\frac{\mu}{2}}$$

is decreasing function for $\lambda \geq 1$ and $\mu \geq 1$ such that $\beta_0 \mu \geq -\delta$.

Lemma B [6]. *Let X be a positive invertible operator and Y be an invertible operator. For any real number λ ,*

$$(YXY^*)^\lambda = YX^{\frac{1}{2}} (X^{\frac{1}{2}} Y^* Y X^{\frac{1}{2}})^{\lambda-1} X^{\frac{1}{2}} Y^*.$$

3 Equivalence Relation Associated with

Theorem 1.

We show the following equivalence relation between Theorem 1 and related operator inequalities.

Theorem 2. *The following (i),(ii),(iii) and (iv) hold and follow from each other.*

(i) *If $A \geq B \geq C \geq 0$ with $A > 0$ and $B > 0$, then for each $t \in [0, 1]$ and $p \geq 1$,*

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \text{ holds for } r \geq t \text{ and } s \geq 1.$$

(ii) *If $A \geq B \geq C \geq 0$ with $A > 0$ and $B > 0$, then for each $1 \geq q \geq t \geq 0$ and $p \geq q$,*

$$A^{q-t+r} \geq \{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} \text{ holds for } r \geq t \text{ and } s \geq 1.$$

(iii) *If $A \geq B \geq C \geq 0$ with $A > 0$ and $B > 0$, then for each $t \in [0, 1]$ and $p \geq 1$,*

$$F_{p,t}(A, B, C, r, s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing function for $r \geq t$ and $s \geq 1$.

(iv) *If $A \geq B \geq C \geq 0$ with $A > 0$ and $B > 0$, then for each $t \in [0, 1]$, $q \geq 0$ and $p \geq t$,*

$$G_{p,q,t}(A, B, r, s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^pB^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing function for $r \geq t$ and $s \geq 1$ such that $(p-t)s \geq q-t$.

We remark that Theorem 2 is an extension of [10], a proof of (i) of Theorem 2 is in [Proposition 4.1, 17], one page proof of (i) by using Theorem G itself is in [8], and also mean theoretic proof of (i) is in [3].

4 Satellite Inequalities.

As simple applications of Theorem 1 and Theorem 2, we show the following satellite inequalities.

Theorem 3. If $A \geq B \geq C > 0$, then the following inequalities (i) and (ii) hold for each $t \in [0, 1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$:

$$\begin{aligned}
 \text{(i)} \quad & B^{\frac{t}{2}} C^{\frac{-r}{2}} \{C^{\frac{r}{2}} (B^{\frac{-t}{2}} A^p B^{\frac{-t}{2}})^s C^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} C^{\frac{-r}{2}} B^{\frac{t}{2}} \\
 & \geq B^{\frac{t}{2}} C^{\frac{-t}{2}} \{C^{\frac{t}{2}} (B^{\frac{-t}{2}} A^p B^{\frac{-t}{2}})^s C^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}} C^{\frac{-t}{2}} B^{\frac{t}{2}} \\
 & \geq A \geq B \geq C \\
 & \geq B^{\frac{t}{2}} A^{\frac{-t}{2}} \{A^{\frac{t}{2}} (B^{\frac{-t}{2}} C^p B^{\frac{-t}{2}})^s A^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}} A^{\frac{-t}{2}} B^{\frac{t}{2}} \\
 & \geq B^{\frac{t}{2}} A^{\frac{-r}{2}} \{A^{\frac{r}{2}} (B^{\frac{-t}{2}} C^p B^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} A^{\frac{-r}{2}} B^{\frac{t}{2}}. \\
 \text{(ii)} \quad & B^{\frac{t}{2}} C^{\frac{-r}{2}} \{C^{\frac{r}{2}} (B^{\frac{-t}{2}} A^p B^{\frac{-t}{2}})^s C^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} C^{\frac{-r}{2}} B^{\frac{t}{2}} \\
 & \geq B^{\frac{t}{2}} C^{\frac{-r}{2}} (C^{\frac{r}{2}} B^{\frac{-t}{2}} A^p B^{\frac{-t}{2}} C^{\frac{r}{2}})^{\frac{1+r-t}{p+r-t}} C^{\frac{-r}{2}} B^{\frac{t}{2}} \\
 & \geq A \geq B \geq C \\
 & \geq B^{\frac{t}{2}} A^{\frac{-r}{2}} (A^{\frac{r}{2}} B^{\frac{-t}{2}} C^p B^{\frac{-t}{2}} A^{\frac{r}{2}})^{\frac{1+r-t}{p+r-t}} A^{\frac{-r}{2}} B^{\frac{t}{2}} \\
 & \geq B^{\frac{t}{2}} A^{\frac{-r}{2}} \{A^{\frac{r}{2}} (B^{\frac{-t}{2}} C^p B^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} A^{\frac{-r}{2}} B^{\frac{t}{2}}.
 \end{aligned}$$

Corollary 4. If $A \geq B > 0$, then the following inequalities (i) and (ii) hold for each $t \in [0, 1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$:

$$\begin{aligned}
 \text{(i)} \quad & B^{\frac{-(r-t)}{2}} \{B^{\frac{r}{2}} (B^{\frac{-t}{2}} A^p B^{\frac{-t}{2}})^s B^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} B^{\frac{-(r-t)}{2}} \\
 & \geq \{B^{\frac{t}{2}} (B^{\frac{-t}{2}} A^p B^{\frac{-t}{2}})^s B^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+r}} \\
 & \geq A \geq B \\
 & \geq \{A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+r}} \\
 & \geq A^{\frac{-(r-t)}{2}} \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} A^{\frac{-(r-t)}{2}} \\
 \text{(ii)} \quad & B^{\frac{-(r-t)}{2}} \{B^{\frac{r}{2}} (B^{\frac{-t}{2}} A^p B^{\frac{-t}{2}})^s B^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} B^{\frac{-(r-t)}{2}} \\
 & \geq B^{\frac{-(r-t)}{2}} (B^{\frac{r-t}{2}} A^p B^{\frac{r-t}{2}})^{\frac{1+r-t}{p+r-t}} B^{\frac{-(r-t)}{2}} \\
 & \geq A \geq B \\
 & \geq A^{\frac{-(r-t)}{2}} (A^{\frac{r-t}{2}} B^p A^{\frac{r-t}{2}})^{\frac{1+r-t}{p+r-t}} A^{\frac{-(r-t)}{2}} \\
 & \geq A^{\frac{-(r-t)}{2}} \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} A^{\frac{-(r-t)}{2}}.
 \end{aligned}$$

Corollary 5. *If $A \geq B > 0$, then the following inequality holds for $p \geq 1$ and $r \geq 0$*

$$B^{-\frac{r}{2}} (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1+r}{p+r}} B^{-\frac{r}{2}} \geq A \geq B \geq A^{-\frac{r}{2}} (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}} A^{-\frac{r}{2}}.$$

5 M.Uchiyama's Result via Theorem A.

The following result is contained in Theorem 3.4 of [17].

Theorem V. *Let A and B be both positive invertible operators. Also let a, b , and c be positive real numbers and d a real number. Define $F(r, s)$ and $G(r, s)$ by*

$$F(r, s) = A^{\frac{r}{2}} (A^{-\frac{r}{2}} B^s A^{-\frac{r}{2}})^{\frac{r}{r+sc}} A^{\frac{r}{2}} \quad \text{for } r > 0, s > 0$$

and

$$G(r, s) = A^{\frac{r}{2}} (A^{-\frac{r}{2}} B^s A^{-\frac{r}{2}})^{\frac{r+d}{r+sc}} A^{\frac{r}{2}} \quad \text{for } r > 0, s > 0 \text{ with } 0 \leq \frac{r+d}{r+sc} \leq 1.$$

Let $a > 0$, $b > 0$ and $-a \leq d \leq bc$. Then for $r_2 \geq r_1 \geq a$ and $s_2 \geq s_1 \geq b$ the following hold:

- (a) *if $F(a, b) \leq 1$, then $G(r_2, s_2) \leq G(r_1, s_1)$*
- (b) *if $F(a, b) \geq 1$, then $G(r_2, s_2) \geq G(r_1, s_1)$.*

On the other hand, in Theorem A replacing A by A^{β_0} and B by B^{α_0} , then we have the following result in [12].

Corollary C. *Let A and B be positive invertible operators on a Hilbert space satisfying*

$$A^{\beta_0} \geq (A^{\frac{\beta_0}{2}} B^{\alpha_0} A^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0 + \beta_0}} \quad \text{for fixed } \alpha_0 > 0 \text{ and } \beta_0 > 0.$$

Then for any fixed $\delta \geq -\beta_0$,

$$f(\alpha, \beta) = A^{-\frac{\beta}{2}} (A^{\frac{\beta}{2}} B^{\alpha} A^{\frac{\beta}{2}})^{\frac{\delta + \beta}{\alpha + \beta}} A^{-\frac{\beta}{2}}$$

is decreasing function of α and β such that $\alpha \geq \max\{\delta, \alpha_0\}$ and $\beta \geq \beta_0$.

We can give a proof of Theorem V via Corollary C.

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